Unified description of quantum particles and electromagnetic and elastic waves in multilayers

1995 J. Phys.: Condens. Matter 7 5491

(http://iopscience.iop.org/0953-8984/7/28/007)

J. Phys.: Condens. Matter 7 (1995) 5491–5506. Printed in the UK

Unified description of quantum particles and electromagnetic and elastic waves in multilayers

G Monsivais†, F García-Moliner and V R Velasco
Instituto de Ciencia de Materiales, CSIC, Serrano 123, 28006 Madrid, Spain

Received 1 March 1995, in final form 11 May 1995

Abstract. Some relevant mathematical properties of quantum particles, polarized electromagnetic waves and elastic shear horizontal waves in multilayer systems are discussed in a unified way. The time-independent wave equations describing these problems are isomorphic for the three cases, but the matching boundary conditions are different. These are also described in common mathematical form by means of appropriate quantities.

Phenomenological concepts and parameters of quantum scattering theory are thus related to different transfer matrices often used for both quantum and classical systems. All multilayers are shown to have the same statistical properties beyond characteristic lengths established by an application of the central limit theorem to the different problems. A Poincaré map representation which often has practical advantages for numerical computation is also set up in a mathematical form common to all cases.

1. Introduction

Although general analogies between electromagnetic waves, elastic waves and quantum particles—Schrödinger equation—have long been noted in the literature [1–3] the study of wave propagation through a multilayer structure is usually carried out in different ways for different types of wave. In particular, basic concepts of scattering theory [2, 4], which are always used to study the propagation of a quantum particle, are seldom employed for other types of wave. The study of the elastic wave propagation in layered media has been an important subject in seismology, and a good summary can be found in [5]. Some different and more recent approaches can be found in [6–8]. We shall restrict our study to the simpler cases in order to fully develop the connections between the different problems. In this paper we study the quantum particle (QP), shear elastic waves (SEW) and polarized electromagnetic waves (EM-S or EM-P) in a unified way. This puts the two classical problems in a frame in which one can make full use of concepts which have proved very useful in quantum mechanical scattering theory.

The differences between these cases lie in the matching boundary conditions at the interfaces but their role can be described in a unified form—section 2—and related to different concepts of *transfer matrices* that one can define. These in turn are related in different ways to phenomenological and scattering theoretic concepts. These relationships are studied in section 3, which includes also a discussion of the statistical aspects.

On the other hand, a Poincaré map [9] representation is known to be useful as a basis for an algorithm for doing numerical calculations. It will be seen in section 4 that starting from

† Permanent address: Instituto de Física, UNAM, Apartado Postal 20-364, 01000 México, DF, Mexico.

0953-8984/95/285491+16$19.50

one of the transfer matrices an appropriate Poincaré map representation can be easily set up which provides a computational algorithm to study particle or wave propagation through a multilayer structure.

2. General formulation and matching boundary conditions

We consider a multilayer structure with interfaces at positions z_n where contiguous media match, with medium $(n-1)$ on the left and medium n on the right of z_n. We shall study a piecewise homogeneous system in which all material parameters—effective mass (m^*), average potential (V), density (ρ), shear rigidity modulus (μ) or dielectric constant (ϵ)—are piecewise constant and vary in a stepwise manner, so each constituent slab is a homogeneous material by itself.

First consider an infinite bulk homogeneous medium and an amplitude of the form

$$e^{i(\boldsymbol{Q}\cdot\boldsymbol{x}-\omega t)}a(z) \tag{1}$$

resulting from a 2D Fourier transform with 2D wavevector $\boldsymbol{Q}$ in the $\boldsymbol{x}$ or (x, y) plane, perpendicular to the z direction. The corresponding planar projection of the differential equation

$$\left(\frac{d^2}{dz^2}+K^2\right)a(z)=0 \tag{2}$$

holds for the following cases.

(i) QP: the Schrödinger equation, with wavefunction

$$\Psi(\boldsymbol{r}, t)=e^{i(\boldsymbol{Q}\cdot\boldsymbol{x}-\omega t)}\psi(z) \qquad a(z)=\psi(z) \tag{3}$$

and

$$K=\sqrt{\frac{2m^*}{\hbar^2}(E-V)-Q^2}. \tag{4}$$

(ii) SEW: shear 'horizontal' elastic waves, with vibration amplitude in the y direction,

$$\boldsymbol{U}=(0, U, 0) \qquad U(\boldsymbol{r}, t)=e^{i(\boldsymbol{Q}\cdot\boldsymbol{x}-\omega t)}u(z) \qquad a(z)=u(z) \tag{5}$$

and

$$K=\sqrt{\frac{\omega^2}{v^2}-Q^2} \qquad v=\sqrt{\frac{\mu}{\rho}}=\text{shear wave velocity}. \tag{6}$$

(iii) Polarized electromagnetic waves. This case requires some comment. First we consider the 2D Fourier transforms of the electric $(\mathcal{E})$ and magnetic $(\mathcal{B})$ fields:

$$\mathcal{E}(\boldsymbol{r}, t)=e^{i(\boldsymbol{Q}\cdot\boldsymbol{x}-\omega t)}\boldsymbol{E}(z) \qquad \mathcal{B}(\boldsymbol{r}, t)=e^{i(\boldsymbol{Q}\cdot\boldsymbol{x}-\omega t)}\boldsymbol{B}(z). \tag{7}$$

By eliminating either $\mathcal{E}$ or $\mathcal{B}$ from Maxwell's equations, the 2D Fourier-transformed differential equation for the remaining field is also (2) with

$$K=\sqrt{\epsilon\frac{\omega^2}{c^2}-Q^2} \tag{8}$$

but the amplitude is then the *vector* $\boldsymbol{E}(z)$ or $\boldsymbol{B}(z)$. Thus, all the components of these vectors satisfy the scalar differential equation (2) and therefore each one of them has the form

$$a(z)=a^+e^{iKz}+a^-e^{-iKz}. \tag{9}$$

However, in the S-polarized case (EM-S) the amplitudes are

$$\mathcal{E} = (0, \mathcal{E}_y, 0) \qquad \mathcal{B} = (\mathcal{B}_x, 0, \mathcal{B}_z) \tag{10}$$

and, since $\mathcal{B}$ can be obtained from curl $\mathcal{E}$, the three non-vanishing components are related by means of

$$(a^\pm)_{Ey} = \mp\frac{\omega}{cK}(a^\pm)_{Bx} = \frac{\omega}{cQ}(a^\pm)_{Bz} \tag{11}$$

where $(a^\pm)_{Ey}$ are the coefficients defined in (9) associated with Ey, etc. Therefore, for the calculation it is sufficient to work with only one component. Alternatively, if one wants to work with the coefficient associated with the total field, i.e. $(a^\pm)_{\mathcal{E}} \equiv \sqrt{(a^\pm)_{\mathcal{E}} \cdot (a^\pm)_{\mathcal{E}}}$ or $(a^\pm)_{\mathcal{B}} \equiv \sqrt{(a^\pm)_{\mathcal{B}} \cdot (a^\pm)_{\mathcal{B}}}$, where $(a^\pm)_{\mathcal{E}} = ((a^\pm)_{Ex}, (a^\pm)_{Ey}, (a^\pm)_{Ez})$, etc, one finds

$$(a^\pm)_{Ey} = (a^\pm)_{\mathcal{E}} = \pm\frac{1}{\sqrt{\epsilon}}(a^\pm)_{\mathcal{B}}. \tag{12}$$

Likewise, in the P-polarized case (EM-P) the amplitudes are

$$\mathcal{E} = (E_x, 0, E_z) \qquad \mathcal{B} = (0, B_y, 0) \tag{13}$$

and, again, it suffices to work with only one component since

$$(a^\pm)_{B_y} = \pm\frac{\omega\varepsilon}{cK}(a^\pm)_{E_x} = -\frac{\omega\varepsilon}{cQ}(a^\pm)_{Ez} \tag{14}$$

which follows from the Maxwell equation for curl $\mathcal{B}$. Furthermore

$$(a^\pm)_{B_y} = (a^\pm)_{\mathcal{B}} = \sqrt{\varepsilon}(a^\pm)_{\mathcal{E}}. \tag{15}$$

Now consider an interface, say at $z = 0$, where two media match and denote

$$\Delta a = a(+0) - a(-0) \qquad \Delta a' = a'(+0) - a'(-0) \qquad a' = \frac{\mathrm{d}a}{\mathrm{d}z}. \tag{16}$$

In general these functions may be discontinuous, so that Δa and $\Delta a'$ may not vanish. However, from the physics of the problems it is easy to identify factors Ω and Π such that

$$\Delta(\Omega a) = 0 \qquad \Delta(\Pi a') = 0. \tag{17}$$

In fact these equalities express the matching boundary conditions at the interface. All the cases under study can then be described in a mathematically identical form in terms of a, Ω, a' and Π, given in table 1. The factors Ω and Π are of course different for the different components of the electromagnetic field. However, it is easy to prove that for the S-polarized case they satisfy

$$(\Omega\Pi K)_{Ey}|(a^\pm)_{Ey}|^2 = \frac{\omega^2}{c^2}(\Omega\Pi K)_{Bx}|(a^\pm)_{Bx}|^2 = \frac{\omega^2}{c^2Q^2}(\Omega\Pi K)_{Bz}|(a^\pm)_{Bz}|^2 \tag{18}$$

and

$$|(a^\pm)_{Ey}|^2 = |(a^\pm)_{\mathcal{E}}|^2 = |\mathcal{E}^\pm|^2 \tag{19}$$

where $(\Omega\Pi K)_F$ represents the values of $(\Omega\Pi K)$ for the component F (E_y, B_x or B_z) of the electromagnetic field. Similarly, for the P-polarized case they satisfy

$$(\Omega\Pi K)_{By}|(a^\pm)_{By}|^2 = \frac{\omega^2}{c^2}(\Omega\Pi K)_{Ex}|(a^\pm)_{Ex}|^2 = \frac{\omega^2}{c^2Q^2}(\Omega\Pi K)_{Ez}|(a^\pm)_{Ez}|^2 \tag{20}$$

and

$$|(a^\pm)_{By}|^2 = |(a^\pm)_{\mathcal{B}}|^2 = |\mathcal{B}^\pm|^2. \tag{21}$$

We shall now study the form of the amplitudes for the multilayer structure and the transfer matrices which can be defined from them.

Table 1. Continuities and discontinuities at an interface at $z = 0$. $\Delta a \equiv a(+) - a(-)$; $\Delta a' \equiv a'(+) - a'(-)$; $\pm$ indicate values at ± 0. Ω $\Delta(\Omega a) \equiv \Omega(+)a(+) - \Omega(-)a(-) = 0$; $\Delta(\Pi a') \equiv \Pi(+)a'(+) - \Pi(-)a'(-) = 0$.

	QP	SEW	EM-S	EM-S	EM-S	EM-P	EM-P
a	ψ	u	E_y	B_x	B_z	E_x	E_z
Δa	0	0	0	0	0	0	$-[\Delta\epsilon/\epsilon(-)]E_z(+)$
Ω	1	1	1	1	1	1	ϵ
$\Delta a'$	$[\Delta m/m(+)]\psi'(+)$	$-[\Delta\mu/\mu(-)]u'(+)$	0	$-\mathrm{i}(\omega/c)\Delta\epsilon\, E_y$	0	$-\mathrm{i}Q[\Delta\epsilon/\epsilon(-)]E_z(+)$	0
Π	$1/m$	μ	1	$1/K^2$	1	ϵ/K^2	1

3. Transfer matrices, phenomenological properties and scattering theoretic concepts

We now consider the multilayer system. Each constituent medium is a slab of a different homogeneous medium n which, as a *bulk* homogeneous medium, would be described by (1)–(15) with corresponding values of m_n^*, V_n, ρ_n, μ_n or ϵ_n. Each interface is at $z = z_n$ and at each one (16), (17) and table 1 are correspondingly applied, with $a(\pm)$ meaning $a(z_n \pm 0)$, etc, in an obvious way.

In domain n, contained between z_n and z_{n+1}, the form of $a(z)$ is

$$a(z) = a_n(z) = a_n^+ e^{iK_n z} + a_n^- e^{-iK_n z}. \tag{22}$$

Now, define the following 2-vectors:

$$\alpha(z) = \begin{bmatrix} a(z) \\ a'(z) \end{bmatrix} \qquad \beta(z) = \begin{bmatrix} \Omega(z)a(z) \\ \Pi(z)a'(z) \end{bmatrix} \qquad \gamma_n(z) = \begin{bmatrix} a_n^+ e^{iK_n z} \\ a_n^- e^{-iK_n z} \end{bmatrix}$$
$$\bar{a}_n = \begin{bmatrix} a_n^+ \\ a_n^- \end{bmatrix} \qquad \bar{A}_n = \Lambda_n^{1/2}\bar{a}_n = \begin{bmatrix} A_n^+ \\ A_n^- \end{bmatrix} \tag{23}$$

where we have defined

$$\Lambda_n = K_n \Omega_n \Pi_n. \tag{24}$$

These 2-vectors are variously used according to convenience. Thus $\alpha(z)$ is practical when $a(z)$ and $a'(z)$ are continuous, while $\beta(z)$ is useful when there are discontinuities, as $\beta(z)$ is always continuous; $\gamma_n(z)$ displays the two components of (22) separately while $\bar{a}_n$ displays the two coefficients of (22) separately and $\bar{A}_n$ is more convenient if one seeks a connection with a scattering theoretic S matrix analysis. The process of integration of the differential equation involves of course all the matching boundary conditions at the interfaces as well as the external or asymptotic boundary conditions outside the multilayer structure, but it can also be partly viewed as the transfer of any of the 2-vectors defined in (23) and this defines different transfer matrices, depending on the choice of the 2-vector to be transferred. The one associated with $\alpha(z)$ has been fully discussed in [10] and further in [11] where, under the name *full transfer matrix*, it has been related to the propagator and to Green function matching techniques. This analysis has also been extended to more complicated situations when the constituent slabs are not necessarily homogeneous or when one deals with a differential system [12], but this is outside the scope of the present study which is concerned with one differential equation and piecewise homogeneous structures. Among the different transfer matrices we shall concentrate mainly on those associated with $\bar{a}_n$ and $\bar{A}_n$, as these will be seen to relate directly to reflection and transmission coefficients, which allows for a direct contact with the scattering matrix [13].

We define the matrix $D(z)$ relating α and β

$$\beta(z) = D(z) \cdot \alpha(z) \qquad D(z) = \begin{Vmatrix} \Omega(z) & 0 \\ 0 & \Pi(z) \end{Vmatrix}. \tag{25}$$

Then, at a given interface, from the continuity of β and the definition of D applied to the two media

$$\alpha(+) = \mathcal{T} \cdot \alpha(-) \qquad \mathcal{T} = D^{-1}(+) \cdot D(-). \tag{26}$$

Thus $\mathcal{T}$ is a *matching transfer matrix*, which transfers $\alpha(z)$ across a matching interface from its $-$ side to its $+$ side. If the interface separates medium $(n-1)$ on its left from medium n on its right, then the corresponding $\mathcal{T}$ is

$$\mathcal{T}_{n,n-1} = D_n^{-1} \cdot D_{n-1}. \tag{27}$$

Now apply (22) to $\bar{a}_n(z)$, letting $z \to z_n + 0$, and to $\bar{a}_{n-1}(z)$, letting $z \to z_n - 0$. Then

$$\alpha(z_n + 0) = B_n \cdot N_n \cdot \bar{a}_n \qquad B_n = \begin{Vmatrix} 1 & 1 \\ \mathrm{i}K_n & -\mathrm{i}K_n \end{Vmatrix} \qquad N_n = \begin{Vmatrix} \mathrm{e}^{\mathrm{i}K_n z_n} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}K_n z_n} \end{Vmatrix} \tag{28}$$

and

$$\begin{aligned} &\alpha(z_n - 0) = B_{n-1} \cdot N_{n-1} \cdot \Delta_{n-1} \cdot \bar{a}_{n-1} \\ &\Delta_{n-1} = \begin{Vmatrix} \mathrm{e}^{\mathrm{i}K_{n-1}(z_n - z_{n-1})} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}K_{n-1}(z_n - z_{n-1})} \end{Vmatrix} \end{aligned} \tag{29}$$

which yield

$$\bar{a}_n = M_{n,n-1}\bar{a}_{n-1} \qquad M_{n,n-1} = N_n^{-1} \cdot B_n^{-1} \cdot D_n^{-1} \cdot D_{n-1} \cdot B_{n-1} \cdot N_{n-1} \cdot \Delta_{n-1}. \tag{30}$$

Evaluation of this matrix yields the result

$$\begin{aligned} &M_{n,n-1} = \frac{1}{2} \begin{Vmatrix} (\sigma + \zeta K_{n-1}/K_n)\,\mathrm{e}^{-\mathrm{i}\lambda^-} & (\sigma - \zeta K_{n-1}/K_n)\,\mathrm{e}^{-\mathrm{i}\lambda^+} \\ (\sigma - \zeta K_{n-1}/K_n)\,\mathrm{e}^{\mathrm{i}\lambda^+} & (\sigma + \zeta K_{n-1}/K_n)\,\mathrm{e}^{\mathrm{i}\lambda^-} \end{Vmatrix} \\ &\sigma = \frac{\Omega_{n-1}}{\Omega_n} \qquad \zeta = \frac{\Pi_{n-1}}{\Pi_n} \qquad \lambda^\pm = (K_n \pm K_{n-1})z_n. \end{aligned} \tag{31}$$

Table 2 gives σ and ζ for the different physical problems listed in table 1.

Table 2. The parameters σ and ζ of (31) for the physical problems listed in table 1.

	QP	SEW	EM-S	EM-S	EM-S	EM-P	EM-P	EM-P
a	ψ	u	E_y	B_x	B_z	E_x	E_z	B_y
σ	1	1	1	1	1	1	$\epsilon_{n-1}/\epsilon_n$	1
ζ	m_n/m_{n-1}	μ_{n-1}/μ_n	1	K_n^2/K_{n-1}^2	1	$K_n^2\epsilon_{n-1}/K_{n-1}^2\epsilon_n$	1	$\epsilon_n/\epsilon_{n-1}$

Note that from equation (12) it follows that the transfer matrix for $((a_n^+)_\mathcal{E}, (a_n^-)_\mathcal{E})$ in S polarization is equal to the transfer matrix for $((a_n^+)_{Ey}, (a_n^-)_{Ey})$. Similarly, the transfer matrix for $((a_n^+)_\mathcal{B}, (a_n^-)_\mathcal{B})$ in P polarization is the same as the one for $((a_n^+)_{By}, (a_n^-)_{By})$ (see equation (15)).

We now concentrate on the study of wave propagation allowed in all the constituent media, when all K_n are real. It is then easily proved (appendix) that

$$\begin{aligned} &(M_{n,n-1})_{11} = (M_{n,n-1})_{22}^* \qquad (M_{n,n-1})_{21} = (M_{n,n-1})_{12}^* \\ &\det(M_{n,n-1}) = \frac{\Lambda_{n-1}}{\Lambda_n} \neq 1. \end{aligned} \tag{32}$$

Let M be the transfer matrix from $z_L = z_{n-m}$ to $z_R = z_n$ and assume also homogeneous media L/R to the left/right of z_L/z_R then

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = M_{n,n-1} \cdot M_{n-1,n-2} \cdot \ldots \cdot M_{n-m+1,n-m} \cdot M_{n-m,n-m-1} \tag{33}$$

is the matrix which transfers $\bar{a}_L \equiv \bar{a}_{n-m-1}$ to $\bar{a}_R \equiv \bar{a}_n$ for a structure with m layers. Now, the general form of $a_L(z)$ is

$$a_L(z) = \begin{cases} \mathrm{e}^{\mathrm{i}K_L z} + r_L \mathrm{e}^{-\mathrm{i}K_L z} & z \leqslant z_L \\ h_L(z) & z_L \leqslant z \leqslant z_R \\ t_L \mathrm{e}^{\mathrm{i}K_R z} & z_R \leqslant z \end{cases} \tag{34}$$

with $K_L \equiv K_{n-m-1}, K_R \equiv K_n$ and that of $a_R(z)$ is

$$a_R(z) = \begin{cases} t_R e^{-iK_L z} & z \leqslant z_L \\ h_R(z) & z_L \leqslant z \leqslant z_R. \\ e^{-iK_R z} + r_R e^{iK_R z} & z_R \leqslant z \end{cases} \tag{35}$$

This describes the full waves in terms of incident and transmitted waves with corresponding reflection and transmission amplitudes, which can be easily related to M. Indeed, an arbitrary solution with incidence from both sides is

$$a_L^+ a_L(z) + a_R^- a_R(z) = \begin{cases} a_L^+ e^{iK_L z} + (r_L a_L^+ + t_R a_R^-) e^{-iK_L z} & z \leqslant z_L \\ a_L^+ h_L(z) + a_R^- h_R(z) & z_L \leqslant z \leqslant z_R. \\ a_R^- e^{-iK_R z} + (t_L a_L^+ + r_R a_R^-) e^{iK_R z} & z_R \leqslant z \end{cases}$$

Thus the amplitudes a_L^-, a_R^+ of the outgoing waves are

$$a_R^+ = t_L a_L^+ + r_R a_R^- \qquad a_L^- = r_L a_L^+ + t_R a_R^-. \tag{36}$$

Since, on the other hand, by definition

$$\begin{bmatrix} a_R^+ \\ a_R^- \end{bmatrix} = \begin{Vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{Vmatrix} \cdot \begin{bmatrix} a_L^+ \\ a_L^- \end{bmatrix} \tag{37}$$

we have

$$\begin{Vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{Vmatrix} = \frac{1}{t_R} \begin{Vmatrix} (t_R t_L - r_R r_L) & r_R \\ -r_L & 1 \end{Vmatrix}. \tag{38}$$

Furthermore, the Wronskian of any two linearly independent solutions of the differential equation describing the full multilayer structure satisfies the relation

$$\Omega_L \Pi_L W(z_L - 0) = \Omega_R \Pi_R W(z_R + 0) \tag{39}$$

and applying this to the four pairs $\{a_L, a_R\}$, $\{a_L, a_L^*\}$, $\{a_L, a_R^*\}$ and $\{a_R, a_R^*\}$ we find

$$\begin{aligned} &\Lambda_L t_R = \Lambda_R t_L \qquad \Lambda_L[1 - |r_L|^2] = \Lambda_R |t_L|^2 \\ &\Lambda_L r_L t_R^* = -\Lambda_R r_R^* t_L \qquad \Lambda_L |t_R|^2 = \Lambda_R[1 - |r_R|^2]. \end{aligned} \tag{40}$$

Therefore

$$t_L t_R - r_L r_R = \frac{t_R}{t_R^*} \qquad -\frac{r_L}{t_R} = \frac{r_R^*}{t_R^*} \tag{41}$$

whence we obtain the simple relationship

$$\begin{Vmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{Vmatrix} = \begin{Vmatrix} 1/t_R^* & r_R/t_R \\ r_R^*/t_R^* & 1/t_R \end{Vmatrix} \tag{42}$$

between the elements of M and the phenomenological scattering amplitudes of the multilayer structures. We also note, from (32) and (33), that

$$\det M = \frac{\Lambda_L}{\Lambda_R} \tag{43}$$

which equals unity only if the two media outside are equal.

A transfer matrix $\mathcal{M}$ with determinant identically equal to unity which has been used to study a system in an electric field [14] can be defined by

$$\bar{A}_n = \mathcal{M}_{n,n-1} \bar{A}_{n-1} \tag{44}$$

which, by (23), is

$$\mathcal{M}_{n,n-1} = \frac{1}{\sqrt{\det(M_{n,n-1})}} M_{n,n-1}. \tag{45}$$

Then, with all K_n real

$$\begin{aligned} (\mathcal{M}_{n,n-1})_{11} &= (\mathcal{M}_{n,n-1})^*_{22} \\ (\mathcal{M}_{n,n-1})_{21} &= (\mathcal{M}_{n,n-1})^*_{12} \\ \det|\mathcal{M}_{n,n-1}| &= 1. \end{aligned} \tag{46}$$

Thus these matrices have the interesting property that they belong to the SU(1, 1) group. As in (33), the matrix $\mathcal{M}$ which transfers $\bar{A}_L$ to $\bar{A}_R$ is

$$\mathcal{M} = \mathcal{M}_{n,n-1} \cdot \mathcal{M}_{n-1,n-2} \cdot \ldots \cdot \mathcal{M}_{n-m+1,n-m} \cdot \mathcal{M}_{n-m,n-m-1} \tag{47}$$

which also belongs to SU(1, 1) and therefore

$$\det(\mathcal{M}) = 1 \tag{48}$$

as is obvious from the definition (44). Now, we define

$$\tau_R = \sqrt{\det(M)}\, t_R \qquad \tau_L = \frac{t_L}{\sqrt{\det(M)}}. \tag{49}$$

It follows from (43) and the first of (40) that

$$\tau_R = \tau_L. \tag{50}$$

On the other hand, from (41)–(45), (48) and (50)

$$\mathcal{M} = \left\| \begin{matrix} 1/\tau_R^* & r_R/\tau_R \\ r_R^*/\tau_R^* & 1/\tau_R \end{matrix} \right\| \tag{51}$$

which, by (48), implies

$$|r_R|^2 + |\tau_R|^2 = 1. \tag{52}$$

This has an interesting physical meaning which bears out the phenomenological relevance of $\mathcal{M}$.

Consider, for each medium n, the flux in the z direction j_n defined in (A6). Suppose, for instance, that there is only incidence from the right, so that $a_L^+ = 0$. Then the ratio of transmitted to incident flux is

$$T \equiv \frac{\Lambda_L |a_L^-|^2}{\Lambda_R |a_R^-|^2} = \frac{|A_L^-|^2}{|A_R^-|^2} = |\tau_R|^2 \tag{53}$$

as follows from (23), (36), (42) and (48), while the ratio of reflected to incident flux is

$$R \equiv \frac{\Lambda_R |a_R^+|^2}{\Lambda_R |a_R^-|^2} = \frac{|A_R^+|^2}{|A_R^-|^2} = |r_R|^2. \tag{54}$$

Thus (52) expresses the rule

$$T + R = 1. \tag{55}$$

On the other hand, from the first and the third of (40) we have

$$|r_L| = |r_R| \tag{56}$$

and, from (50), it follows that (52) implies

$$|r_L|^2 + |\tau_L|^2 = 1 \tag{57}$$

as is obvious on physical grounds. We note that, although the transfer matrices associated with each component of the electromagnetic field are in general different, all of them give rise to the same T and R as defined in equations (53), (54). In fact, from equations (18), (20) we have that the value of $(\Lambda_R/\Lambda_L)|a_R^+/a_L^+|^2$ is the same for all the components. Furthermore, from equations (19), (21) and since $K_L = K_{n-m-1}$, $K_R = K_n$, we see that T equals the ratio of the transmitted energy flux to the incident energy flux in the z direction, i.e. $T = |\hat{z}\cdot s_R^+|/|\hat{z}\cdot s_L^+|$, where s_n^+ is the Poynting vector. Thus j_n (A6) is the energy flux in the z direction and the same holds for the SEW case, while in the QP case it is the probability density current.

It follows from these results that, while a 2×2 complex matrix has in general eight parameters, the matrix $\mathcal{M}$ of (57) can be expressed in terms of just three parameters, so we can cast $\mathcal{M}$ for the different physical problems here studied, classical and quantum, in terms of the Bargmann parameters ϑ, ν, ϱ [15] as

$$\mathcal{M} = \left\| \begin{array}{cc} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12}^* & \mathcal{M}_{11}^* \end{array} \right\| = \left\| \begin{array}{cc} \sqrt{1+\varrho}\,\mathrm{e}^{-\mathrm{i}(\vartheta+\nu)} & \sqrt{\varrho}\,\mathrm{e}^{-\mathrm{i}(\vartheta-\nu)} \\ \sqrt{\varrho}\,\mathrm{e}^{\mathrm{i}(\vartheta-\nu)} & \sqrt{1+\varrho}\,\mathrm{e}^{\mathrm{i}(\vartheta+\nu)} \end{array} \right\| \tag{58}$$

with $-\pi \leqslant \vartheta, \nu \leqslant \pi$ and $0 \leqslant \varrho \leqslant \infty$. Put

$$r_R = |r_R|\mathrm{e}^{\mathrm{i}\phi_{rR}} \qquad r_L = |r_L|\mathrm{e}^{\mathrm{i}\phi_{rL}} \qquad t_R = |t_R|\mathrm{e}^{\mathrm{i}\phi_{tR}} \qquad r_L = |t_L|\mathrm{e}^{\mathrm{i}\phi_{tL}}. \tag{59}$$

It follows from (49) and (50) that t_L, τ_L, t_R, τ_R all have the same phase ϕ_t $(= \phi_{tR} = \phi_{tL})$, and, from (40)

$$\phi_{rR} + \phi_{rL} = 2\phi_t + \pi. \tag{60}$$

These results yield the relations

$$\vartheta = -\tfrac{1}{2}\phi_{rR} \qquad \nu = \frac{\pi}{2} - \tfrac{1}{2}\phi_{rL} \tag{61}$$

and

$$\varrho = \frac{R}{T}. \tag{62}$$

This extends the Landauer formula [16] for the dimensionless resistance ϱ of the QP case to the classical EM and SEW cases.

The latter requires a trivial clarification when the multilayer structure terminates at the vacuum. Suppose an incident wave in medium $L = n-m-1$ meets the multilayer structure with the vacuum in medium R, on the right, where there is no elastic wave amplitude.

The total reflectivity then has unit amplitude and the only unknown is its phase. From (29), formally

$$\left[\begin{array}{c} a(z_n - 0) \\ \mu_{n-1}a'(z_n - 0) \end{array} \right] = \left\| \begin{array}{cc} 1 & 0 \\ 0 & \mu_{n-1} \end{array} \right\| \cdot B_{n-1}\cdot N_{n-1}\cdot \Delta_{n-1}\cdot M_{n-1,n-m-1}\cdot \left[\begin{array}{c} a_L^+ \\ a_L^- \end{array} \right] \tag{63}$$

but at the interface with the vacuum the stress $\mu_{n-1}a'(z_n - 0)$ vanishes, whence we obtain the ratio a_L^-/a_L^+.

Having discussed the relationships of the transfer matrices with the phenomenological concepts of scattering theory, it is now interesting to discuss the relationship with alternative scattering matrices. We shall concentrate on those associated with $a_n^\pm$ and $A_n^\pm$, but will later mention the connection with other scattering matrices which can equally be defined. We start by defining S by

$$\left(\begin{array}{c} a_R^+ \\ a_L^- \end{array} \right) = S\cdot \left(\begin{array}{c} a_L^+ \\ a_R^- \end{array} \right) \tag{64}$$

where outgoing amplitudes are obtained, through S, from incoming ones. From (36) and (38)

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \frac{1}{M_{22}} \left\| \begin{matrix} \det(M) & M_{12} \\ -M_{21} & 1 \end{matrix} \right\| = \left\| \begin{matrix} t_L & r_R \\ r_L & t_R \end{matrix} \right\| \tag{65}$$

which in general is *not* unitary. If instead we define the matrix $\mathcal{S}$ which relates $A^{\pm}$ amplitudes, i.e.

$$\begin{bmatrix} A_R^+ \\ A_L^- \end{bmatrix} = \mathcal{S} \begin{bmatrix} A_L^+ \\ A_R^- \end{bmatrix} \tag{66}$$

then

$$\mathcal{S} = \left\| \begin{matrix} S_{11}[\det(M)]^{-1/2} & S_{12} \\ S_{21} & S_{22}[\det(M)]^{1/2} \end{matrix} \right\| = \left\| \begin{matrix} \tau_L & r_R \\ r_L & \tau_R \end{matrix} \right\|. \tag{67}$$

From the above results we find

$$\det\ \mathcal{S} = \frac{\tau_R}{\tau_R^*} = \mathrm{e}^{2\mathrm{i}\phi_t}. \tag{68}$$

Thus, in correspondence with (48)—mathematically—and (55)—physically—$\mathcal{S}$ is a unitary matrix. Its eigenvalues are

$$\mathrm{e}^{\mathrm{i}\delta_1} = (|\tau_L| + \mathrm{i}|r_L|)\mathrm{e}^{\mathrm{i}\phi_t} \qquad \mathrm{e}^{\mathrm{i}\delta_2} = (|\tau_L| - i|r_L|)\mathrm{e}^{\mathrm{i}\phi_t} \tag{69}$$

with $\delta_1 + \delta_2 = 2\phi_t$. We note that if we permute the rows of the matrix defined in (67) (which corresponds to the definition used in [17]), the determinant is equal to $-\tau_L/\tau_L^*$ and the eigenvalues are given by

$$\mathrm{e}^{\mathrm{i}\delta'_{1,2}} = \left(|r_L|\cos\phi \pm \mathrm{i}\sqrt{1 - |r_L|^2\cos^2\phi}\right)\mathrm{e}^{\mathrm{i}\Theta} \tag{70}$$

with $\delta'_1 + \delta'_2 = 2\phi_t + \pi$. On the other hand it is easy to prove that the density of states contained in the scatterer is given by [17]

$$\eta(E) = \frac{1}{2\pi}\frac{\partial(\delta_1 + \delta_2)}{\partial E}. \tag{71}$$

It is also interesting to discuss another scattering matrix, $\mathcal{S}''$, which relates functions, rather than coefficients. For this we note that, since $a_L(z)$ and $a_R(z)$ are linearly independent and $a_L^*(z)$ and $a_R^*(z)$ are also solutions of the differential equations, the latter must be linear combinations of the former. In fact, from the above results we find, after some algebra

$$a_L^*(z) = t_L^* a_R(z) + r_L^* a_L(z) \qquad a_R^*(z) = r_R^* a_R(z) + t_R^* a_L(z) \tag{72}$$

which can be compacted in the form

$$\begin{bmatrix} \Lambda_L^{-1/2} a_L^*(z) \\ \Lambda_R^{-1/2} a_R^*(z) \end{bmatrix} = \mathcal{S}'' \cdot \begin{bmatrix} \Lambda_R^{-1/2} a_R(z) \\ \Lambda_L^{-1/2} a_L(z) \end{bmatrix} \tag{73}$$

with

$$\mathcal{S}'' = \left\| \begin{matrix} \tau_L^* & r_L^* \\ r_R^* & \tau_R^* \end{matrix} \right\|. \tag{74}$$

It is easily seen that $\mathcal{S}''$ is the inverse of $\mathcal{S}$ and its eigenvalues are the complex conjugate of those of (67). The amplitudes $a_L(z)$—see (34)—and $a_R(z)$—see (35)—are incoming state amplitudes consisting of one incident—unperturbed—wave and two scattered waves. In the range of allowed propagations their complex conjugates are outgoing state amplitudes and it is easily seen that these amplitudes satisfy equations of the Lippmann–Schwinger type, as in quantum scattering theory [2, 4].

Now, S'' can be diagonalized by means of the matrix

$$P = \frac{1}{\sqrt{2}} \left\| \begin{array}{cc} e^{-i\phi_{rR}} & -ie^{-i\phi_t} \\ e^{-i\phi_{rR}} & ie^{-i\phi_t} \end{array} \right\| \tag{75}$$

and if we define new amplitudes Φ by

$$\begin{aligned} \left[\begin{array}{c} \Phi_1^{out}(z) \\ \Phi_2^{out}(z) \end{array} \right] &= P \cdot \left[\begin{array}{cc} \Lambda_L^{-1/2} & a_L^*(z) \\ \Lambda_R^{-1/2} & a_R^*(z) \end{array} \right] \\ \left[\begin{array}{c} \Phi_1^{in}(z) \\ \Phi_2^{in}(z) \end{array} \right] &= P \cdot \left[\begin{array}{cc} \Lambda_R^{-1/2} & a_R(z) \\ \Lambda_L^{-1/2} & a_L(z) \end{array} \right] \end{aligned} \tag{76}$$

then (73) reads

$$\left[\begin{array}{c} \Phi_1^{out}(z) \\ \Phi_2^{out}(z) \end{array} \right] = \left\| \begin{array}{cc} e^{-i\delta_1} & 0 \\ 0 & e^{-i\delta_2} \end{array} \right\| \cdot \left[\begin{array}{c} \Phi_1^{in}(z) \\ \Phi_2^{in}(z) \end{array} \right]. \tag{77}$$

Moreover, the relationship between incoming–outgoing amplitudes acquires then a different form displaying the role of the phases. Thus

$$\sqrt{2}\Phi_p^{in}(z) = \begin{cases} -\Lambda_L^{-1/2} \left\{(-1)^{p-1}i\ e^{i(K_L z-\phi_t)} + e^{-i(K_L z-\phi_{rL}+\delta_q)}\right\} & z \leqslant z_L \\ \Lambda_R^{-1/2}\ e^{-i\phi_{rR}} h_R(z) + (-1)^p i\ \Lambda_L^{-1/2} e^{-i\phi_t} h_L(z) & z_L \leqslant z \leqslant z_R \\ \Lambda_R^{-1/2} \left\{e^{-i(K_R z+\phi_{rR})} + (-1)^p i\ e^{i(K_R z+\phi_t-\delta_q)}\right\} & z_R \leqslant z \end{cases} \tag{78}$$

where $q = 2, 1$ when $p = 1, 2$. In this picture the incident and reflected waves have the same amplitude and the phases of the incident waves are directly related to the phases of the coefficients of $a_L(z)$ and $a_R(z)$ in the linear combinations of (76), while the phases of the reflected waves involve also the additional terms $-\delta_q$, which are the phase shifts produced by the entire multilayer structure as a scatterer.

The situation is the opposite for

$$\Phi_p^{out}(z) = e^{-i\delta_p}\Phi_p^{in}(z). \tag{79}$$

By using (60) it is seen that for the $\Phi^{out}(z)$ waves it is the reflected amplitudes that are undistorted, while the incident amplitudes have phase shifts. We stress that (79) holds everywhere, for all z, even inside the multilayer structures. On the other hand, if we define a scattering matrix S''' which relates asymptotic amplitudes, then we find that this is the matrix S of (65) [18]. However, this definition has the disadvantage that is not directly generalizable to the case $K_L \neq K_R$.

Now, the different physical problems here considered have different values of Ω and Π and therefore they have different transfer matrices. Thus some details can be significantly different, as is borne out for instance by the study of Stark ladder resonances in the different cases [19–21]. However, the common mathematical form suggests that for sufficiently large lengths the statistical properties should be the same in all cases as we shall see below. Let p_n denote the value that some characteristic parameter—m_n^*, V_n, v_n or ϵ_n—takes in the constituent slab n. We note that, since disorder leads to a randomization of the phases [22], if we have an ensemble of N disordered systems of length $\mathcal{L}$, all with the same values of p_L and p_R with some distribution of $\{p_j\}(n-1-m < j < n)$, as $\mathcal{L}$ increases—with the number m of layers or 'scatterers' also increasing—the distribution of phases $\phi_t, \phi_{rR}, \phi_{rL}$ tends to spread out. We define the *randomization length* ℓ as the value of $\mathcal{L}$ for which the Bargmann parameters ϑ, ν are uniformly distributed between $-\pi$ and $+\pi$. We denote by ℓ(QP), ℓ(SEW) and ℓ(EM) the value of ℓ for the different physical problems here considered. These values of course could be different, but if

$$\mathcal{L} > \max\{\ell(\text{QP}),\ \ell(\text{SEW}),\ \ell(\text{EM})\} \tag{80}$$

then we expect the statistical properties to be equal in all cases. In fact it has been proved analytically [14, 23] that if $\mathcal{L} > \ell$ then the average properties of the Bargmann parameter ϱ obey a generalization of the central limit theorem in which the distribution $P_m(\mathcal{M})$ of the total matrix $\mathcal{M}$ is independent of the distributions $P_1(\mathcal{M}_{j-1,j})$ of the partial matrices. The generalization is obtained by performing averages over the SU(1, 1) group and it is found that, for the case $p_L = p_R$, the distribution of $\ln(1+\varrho)$ is Gaussian. This has also been obtained in various numerical calculations for the different physical problems here considered [1, 24–26 and references therein]. In the quantum mechanical case the average of $\ln(1+\varrho)$ is usually written in the form [9]

$$\langle \ln(1+\varrho) \rangle = 2\frac{\mathcal{L}}{L_c}. \tag{81}$$

This defines the *localization length* L_c which can therefore be obtained from the easily performed average over a Gaussian distribution.

On the other hand, for the less studied case $P_L \neq P_R$, the distribution of $\ln(1+\varrho)$ is not Gaussian [9, 27] but it is also independent of the distribution $P_1(\mathcal{M}_{n-1,n})$ of the partial matrices $\mathcal{M}_{n-1,n}$ as was also proved analytically [14]. Therefore, the statistical properties of the three systems are again the same.

4. Poincaré maps for a multilayer structure

It has been found that the usual 1D Schrödinger equation admits a Poincaré map representation for some potentials [9, 28–36] including models of nonlinear systems. Basically this is an alternative to the transfer matrix. The Poincaré map method appears to have in practice definite advantages especially when one needs both the wavefunction and its derivative with respect to energy or in the study of nonlinear systems [31, 33]. It is interesting to stress that the Poincaré map representation can equally be set up for the physical problems here considered. The 1D stepwise potential is just a particularly simple case, but a stepwise variation of the effective mass for a particle in a 3D multilayer can also be included and, in keeping with the mathematical isomorphism, the method can equally be applied to the SEW and EM cases. From (28) and (29) we have

$$\begin{aligned}
&\alpha(z_{n+1}+0) = \mathcal{A}_n \cdot \alpha(z_n+0)\\
&\mathcal{A}_n = \left\| \begin{array}{cc} (\Omega_n/\Omega_{n-1})C_n & (1/K_n)(\Omega_n/\Omega_{n-1})S_n \\ -K_n(\Pi_n/\Pi_{n+1})S_n & (\Pi_n/\Pi_{n+1})C_n \end{array} \right\|\\
&C_n = \cos[K_n(z_{n+1}-z_n)]\\
&S_n = \sin[K_n(z_{n+1}-z_n)].
\end{aligned} \tag{82}$$

Changing in this formula $n \to (n-1)$ and rearranging terms

$$\begin{aligned}
&\begin{bmatrix} a'(z_n+0) \\ a'(z_{n-1}+0) \end{bmatrix}\\
&\qquad = \frac{K_{n-1}}{S_{n-1}} \left\| \begin{array}{cc} (\Omega_n\Pi_{n-1}/\Omega_{n-1}\Pi_n)C_{n-1} & -(\Pi_{n-1}/\Pi_n) \\ (\Omega_n/\Omega_{n-1}) & -C_{n-1} \end{array} \right\| \cdot \begin{bmatrix} a(z_n+0) \\ a(z_{n-1}+0) \end{bmatrix}
\end{aligned} \tag{83}$$

which in particular yields a formula for $a'(z_n+0)$ in terms of $a(z_n+0)$ and $a(z_{n-1}+0)$. Using this in (82) we obtain the Poincaré maps

$$\Omega_{n+1}a(z_{n+1}+0) = \Omega_n \left[C_n + \frac{\Pi_{n-1}K_{n-1}}{\Pi_n K_n} \cdot \frac{\Omega_n}{\Omega_{n-1}} \cdot \frac{S_n}{S_{n-1}} \cdot C_{n-1} \right] a(z_n+0)$$

$$
\begin{aligned}
&\quad - \Omega_n \frac{\Pi_{n-1} K_{n-1}}{\Pi_n K_n} \cdot \frac{S_n}{S_{n-1}} a(z_{n-1}+0) \\
\Pi_{n+1} a'(z_{n+1}+0) &= \Omega_n \left[-\frac{\Pi_n K_n}{\Omega_n} S_n + \frac{\Pi_{n-1} K_{n-1}}{\Omega_{n-1}} \cdot \frac{C_{n-1}}{S_{n-1}} \cdot C_n \right] a(z_n+0) \\
&\quad - \Pi_{n-1} K_{n-1} \cdot \frac{C_n}{S_{n-1}} a(z_{n-1}+0).
\end{aligned}
\tag{84}
$$

Similar relationships between the amplitudes evaluated at $z_{n+1}-0$, z_n-0 and $z_{n-1}-0$ can be likewise obtained.

To use the Poincaré map representation between the points z_L and z_R we need to give two values of $a(z)$ which can be given when we have an incoming wave from one side [9], that is, incident and reflected amplitudes on one side and only a transmitted amplitude on the other side—except for the SEW case, which will be presently considered. For incidence from the right, for instance, the amplitude of the left is $a_L^- \exp(-\mathrm{i}K_L z)$, which we evaluate at two different points, $z_{L'}, z_{L''}$ on the left of z_L. Then, by using (84) we obtain $a(z)$ everywhere, including two arbitrarily chosen points $z_{R'}$ and $z_{R''}$ on the right of z_R. To calculate the transmission and reflection coefficients we need the coefficients associated with the wave on the right which, by (28), (83) and (84), are given by

$$
a_R^- = \frac{a(z_{R'})\mathrm{e}^{\mathrm{i}K_R(z_{R'}-z_R)} - a(z_{R''})\mathrm{e}^{\mathrm{i}k_R z_R}}{1-\mathrm{e}^{-2\mathrm{i}K_R(z_{R'}-z_R)}} \qquad a_R^+ = \frac{a(z_R^+) - a_R^- \mathrm{e}^{-\mathrm{i}k_R z_R}}{\mathrm{e}^{\mathrm{i}K_R z_R}} \tag{85}
$$

where we have used the fact that Ω, Π and K are the same for z_R as for $z_{R'}$ and $z_{R''}$, which are two arbitrarily chosen points in the same medium R. Similar formulae for $a_{L'}^+$ and $a_{L'}^-$ respectively hold for incidence from the left with the changes $(z_R, z_{R'}, z_{R''}) \to (z_{L'}, z_{L''}, z_{L'''})$ and $K_R \to -K_L$ except in $(z_{R'} - z_R)$ that must be changed by $-(z_L - z_{L'})$. In the SEW case, for incidence from the right and vacuum on the left there is no amplitude for $z < z_L$. However, since the stress $\mu_{n-m}a'(z_{n-m}+0)$ vanishes, from a similar equation to the second equality of (84), with the appropriate and evident changes we can calculate $a(z_{n-m+2}+0)$ as a function of $a(z_{n-m+1}+0)$ and, using these two values in the first equality of (84), we can carry out the process as in the other cases.

Thus a Poincaré map representation can be set up in all cases and used as an algorithm for doing practical calculations.

5. Conclusions

Since the different physical problems here considered can be cast in the same mathematical form, this isomorphism can be used to set up one single formal treatment in which different concepts of transfer matrices are related to various scattering matrices that one can define, and to phenomenological concepts and parameters of quantum scattering theory, which is thus equally applicable to the classical wave problems under consideration. The distinguishing feature of these various cases lies in the matching boundary conditions at the interfaces which, however, can also be cast in the same mathematical form in terms of the quantities Ω and Π given in table 1.

Numerical calculations can be carried out with the same algorithm for all these cases by appropriate transliteration, the input consisting of the corresponding values of the parameters entering each problem and characterizing the materials under study. This algorithm can be based on the use of transfer matrices but an interesting alternative can be based on Poincaré map representation which often has practical advantages and can also be set up in a mathematical form common to all cases.

Such a completely unified treatment is only possible for the case of stepwise varying parameters. This is a simple model which, nevertheless, is very often useful in practice. For more general models it would not be possible to obtain a completely unified treatment, but part of the analysis here presented could be extended. For instance, the relationship between the full transfer matrix and the propagator is quite general and holds for any differential system [11, 12], which may include, say, a many-band envelope function model, a full study of the 3D elastic vector field or a complete analysis of EM waves including coupling between longitudinal and transverse waves. Moreover, the propagator can be easily related to the phenomenological parameters of scattering—i.e. reflection and/or transmission—theory [12]. It would be interesting to pursue such attempts as partial extensions of the present analysis.

Another interesting aspect concerns the statistical properties of multilayer structures, discussed in section 3. The fact that one can identify transfer matrices belonging to the SU(1, 1) group allows one to apply the generalization of the central limit theorem for this group, thus establishing that, for lengths larger than the maximum randomization length of the different problems here considered, the statistical properties of the multilayer structure are the same in all cases and, in each one of them, for lengths larger than its own randomization length, the total transfer matrix is independent of the detailed distribution of transfer matrices for the different layers, i.e. of the details of the multilayer structures. These properties are likely to hold also for more general models, which would constitute another interesting task, as well as for more detailed studies, in any case, of the randomization lengths for the different physical problems.

Acknowledgments

One of us (GM) is especially indebted to the Spanish Ministry of Education and Science for the award of a sabbatical visiting professorship, during which part of this work was done. This work was partially supported by the Spanish CICYT under grant No PB93-1251 and the European Community through contract No CI1*-CT93-0225.

Appendix. Symmetries of the transfer matrix

Consider the form of $a(z)$ given in (22). For allowed propagations, with all K_n real, the complex conjugate

$$a^*(z) = (a_n^-)^* e^{iK_n z} + (a_n^+)^* e^{-iK_n z} \tag{A1}$$

is also a solution of the same differential equation and therefore the components of the 2-vector $\bar{a}_n^*$ are also transferred by the same transfer matrix $M_{n,n-1}$ as the ones of $\bar{a}_n$ are in (30). Thus

$$\begin{bmatrix} (a_n^-)^* \\ (a_n^+)^* \end{bmatrix} = M_{n,n-1} \cdot \begin{bmatrix} (a_{n-1}^-)^* \\ (a_{n-1}^+)^* \end{bmatrix}. \tag{A2}$$

Taking the complex conjugate, and denoting in this appendix for simplicity $(M_{n,n-1})_{ij} = M_{ij}$

$$\begin{bmatrix} a_n^- \\ a_n^+ \end{bmatrix} = \begin{Vmatrix} M_{11}^* & M_{12}^* \\ M_{21}^* & M_{22}^* \end{Vmatrix} \cdot \begin{bmatrix} a_{n-1}^- \\ a_{n-1}^+ \end{bmatrix} \tag{A3}$$

and rearranging:

$$\begin{bmatrix} a_n^+ \\ a_n^- \end{bmatrix} = \begin{Vmatrix} M_{22}^* & M_{21}^* \\ M_{12}^* & M_{11}^* \end{Vmatrix} \cdot \begin{bmatrix} a_{n-1}^+ \\ a_{n-1}^- \end{bmatrix}. \tag{A4}$$

Hence, from (A4) and (30):

$$M_{11} = M_{22}^* \qquad M_{12} = M_{21}^* \tag{A5}$$

which proves (32).

Now define the flux in the z direction. For medium n this can be cast in the form (omitting the terms with null divergence)

$$j_n = C\Lambda_n \left[|a_n^+|^2 - |a_n^-|^2\right]. \tag{A6}$$

Here j_n is the probability current density, elastic energy flux or Poynting's vector for the QP, SEW or EM waves respectively in the z direction. In the last case the factor Λ and the amplitudes $a^\pm$ are those pertaining to E_y, B_x or B_z for S polarization or B_y, E_x or E_z for P polarization (see equations (18), (20)). In all cases C is a constant, independent of the medium. We can now cast the type of analysis presented in [37] for the QP in a way which is common to all the physical problems here considered. We note that j_n can be written in the form

$$j_n = C\Lambda_n \bar{a}_n^\dagger \cdot \sigma_z \cdot \bar{a}_n \tag{A7}$$

where σ_z is the Pauli matrix

$$\sigma_z = \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|. \tag{A8}$$

Equating fluxes at $z_n \pm 0$ we have

$$\frac{\Lambda_{n-1}}{\Lambda_n}\sigma_z = M_{n,n-1}^\dagger \cdot \sigma_z \cdot M_{n,n-1} \tag{A9}$$

whence

$$\det(M_{n,n-1}) = \frac{\Lambda_{n-1}}{\Lambda_n}. \tag{A10}$$

Since the total transfer matrix across a multilayer stack is the product of the partial transfer matrices, the formula (43) follows from (A10) on general grounds and, by the same argument we obtain also (48) which is thus seen to follow directly from flux conservation. The results contained in (52)–(55) bear out a complementary aspect of this basic property. This implies in turn that the S matrix of equation (67) is a unitary matrix but not the S matrix of equation (65).

References

[1] Ping Sheng (ed) 1989 *Scattering and Localization of Classical Waves in Random Media* (Singapore: World Scientific)

[2] Newton R G 1982 *Scattering Theory of Waves and Particles* (New York: Springer)

[3] Lieb E H and Mattis D C 1966 *Mathematical Physics in One Dimension* (New York: Academic)
Mattis D C 1993 *The Many-Body Problem (An Encyclopedia of Exactly Solved Models in One Dimension)* (Singapore: World Scientific)

[4] Roman P 1965 *Advanced Quantum Theory* (Reading, MA: Addison-Wesley) pp 286–304

[5] Kennett B L N 1983 *Seismic Wave Propagation in Stratified Media* (Cambridge: Cambridge University Press)

[6] Thomson C, Clarke T and Garmany J 1986 *Geophys. J. R. Astron. Soc.* **86** 675

[7] Tromp J and Snieder R 1989 *Geophys. J.* **96** 447

[8] Jeffryes B P 1993 *Phys. Rev. Lett.* **71** 1119

[9] Soukoulis C M, José J V, Economou E N, and Sheng P 1983 *Phys. Rev. Lett.* **50** 764

[10] Mora M E, Pérez R and Sommers Ch B 1985 *J. Physique* **46** 1021

[11] Pérez-Alvarez R, Rodríguez-Coppola H, Velasco V R and García-Moliner F 1988 *J. Phys. C: Solid State Phys.* **21** 1789
[12] García-Moliner F, Pérez-Alvarez R, Rodríguez-Coppola H and Velasco V R 1990 *J. Phys. A: Math. Gen.* **23** 1405
García-Moliner F and Velasco V R 1992 *Theory of Single and Multiple Interfaces: The Method of Surface Green Function Matching* (Singapore: World Scentific)
[13] Borland R E 1961 *Proc. Phys. Soc.* **78** 926
[14] Flores J, Mello P A and Monsivais G 1987 *Phys. Rev.* B **35** 2144
[15] Bargmann V 1947 *Ann. Math.* **48** 568
[16] Landauer R 1970 *Phil. Mag.* **21** 863
[17] Anderson P W and Lee P A 1980 *Prog. Theor. Phys. Suppl.* **69** 212
[18] Galindo A and Pascual P 1991 *Quantum Mechanics* vols I, II (Berlin: Springer)
[19] Bentosela F, Grecchi V and Zironi F 1982 *J. Phys. C: Solid State Phys.* **15** 7119
[20] Monsivais G, del Castillo M and Claro F 1990 *Phys. Rev. Lett.* **64** 1433
[21] Mateos J L and Monsivais G 1994 *Physica* A **207** 445
[22] Anderson P W, Thouless D J, Abrahams A and Fisher D S 1980 *Phys. Rev.* B **22** 3519
[23] Mello P A 1986 *J. Math. Phys.* **27** 2876
[24] Andereck B S and Abrahams E 1980 *J. Phys. C: Solid State Phys.* **13** L383
[25] Esquivel R and Cocoletzi G H *J. Acoust. Soc. Am.* **95** 1994
[26] Sheng P, White B, Zhang Z Q and Papanicolaou G 1986 *Phys. Rev.* B **34** 4757
Sipe J E, Sheng P, White B and Cohen M H 1988 *Phys. Rev. Lett.* **60** 108
Zhang Z Q and Sheng P 1994 *Phys. Rev.* B **49** 83
[27] Flores J, José J V and Monsivais G 1983 *J. Phys. C: Solid State Phys.* **16** L103
[28] Kvitsinsky A A 1994 *J. Phys. A: Math. Gen.* **27** 215
[29] Cota E, José V and Monsivais G 1989 *Phys. Rev.* B **35** 8929
[30] Bellissard J, Formoso A, Lima R and Testard D 1982 *Phys. Rev.* B **26** 3024
[31] José J V, Monsivais G and Flores J 1985 *Phys. Rev.* B **31** 6906
[32] Sokoloff J B and José J V 1982 *Phys. Rev. Lett.* **49** 334
[33] Cota E, José J V and Monsivais G 1992 *J. Phys. A: Math. Gen.* **25** L57
[34] Wan Y and Soukoulis C M 1990 *Phys. Rev.* A **41** 800
[35] Grabowsky M and Hawrylak P 1990 *Phys. Rev.* B **41** 5783
Hawrylak P, Grabowsky M and Wilson P 1989 *Phys. Rev.* B **40** 6398, 8013
[36] Delyon F, Levy Y and Souilliard B 1986 *Phys. Rev. Lett.* **57** 2010
[37] Erdös P and Herndon C 1982 *Adv. Phys.* **31** 63